Vigorous 3D Angular Resection Model Using Levenberg – Marquardt Method

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Abstract

The resection in 3D space is a common problem in surveying engineering and photogrammetry based on observed distances, angles, and coordinates. This resection problem is nonlinear and comprises redundant observations which is normally solved using the least-squares method in an iterative approach. In this paper, we introduce a vigorous angular based resection method that converges to the global minimum even with very challenging starting values of the unknowns. The method is based on deriving oblique angles from the measured horizontal and vertical angles by solving spherical triangles. The derived oblique angles tightly connected the rays enclosed between the resection point and the reference points. Both techniques of the nonlinear least square adjustment either using the Gauss-Newton or Levenberg – Marquardt are applied in two 3D resection experiments. In both numerical methods, the results converged steadily to the global minimum using the proposed angular resection even with improper starting values. However, applying the Levenberg – Marquardt method proved to reach the global minimum solution in all the challenging situations and outperformed the Gauss-Newton method.

I. INTRODUCTION

Resection in 3D space is a common problem in surveying engineering and photogrammetry to fix the position at an unknown occupied point. This is normally applied either by measuring distances, angles, or other quantities that define the intersected rays in space coming from reference points. Mostly, all the 3D resection models are either nonlinear and subject to the proper starting values or direct linear solutions that cannot handle the redundant observations [1]. Additionally, these resection methods are vulnerable to the inadequate distribution of the points [2]. Conventionally, angular based resection in 3D space is applied by utilizing the horizontal and vertical angles. However, the resection model is highly sensitive to initial values that should not be far by more than 1' to ensure the correct solutions, otherwise, the solution diverges to wrong values [3-5].

Accordingly, this research aims to develop a generic mathematical model for the 3D angular based resection that consider the redundancy in the observations and vigorous even with incorrect starting values. This implies introducing a method that is: geometrically stable, reliable, and converges to the global minimum even with improper starting values.

Recently, [6] introduced a robust resection model that relies on deriving oblique angles out of the measured horizontal and vertical angles between the unknown camera station and the reference points. The nonlinear least-squares method NLLS is further applied to handle the redundant observations using the Gauss-Newton method.

Although, we don't need to calculate the second-order derivative when using the Gauss-Newton this method may not be able to handle highly challenging starting values because of the high residual errors and then better adaptation methods should be used [7].

In this paper, we are going to proceed with the oblique angular resection model and to elaborate on the use of the

Levenberg – Marquardt LM or the damped least-squares method [8, 9]. Currently, Levenberg – Marquardt is a widely used technique for solving the nonlinear least-squares problems and is considered as a mixed method between the Gradient Descent and the Gauss-Newton methods [10] as shown in Fig.1.

The gradient descent method is based on updating the solution and selecting values that make the function value smaller in each iteration. Precisely, the sum of the squared errors is reduced by moving toward the direction of steepest descent and ensures to converge linearly. Whilst the Newton method converges quadratically in the direction of the minimum, but it fails to converge if it is far away from its minima.

To overcome the problem of failed convergence and reaching a robust solution, Levenberg – Marquardt method utilized a modified enhancement to the Gauss-Newton method by depending on the value of an algorithmic parameter λ [11]. This parameter λ is a non-negative damping factor that suppresses the solution by adjusting its value at each iteration to ensure a reduction in the error. According to [12], the least square normal equation matrix becomes nearly diagonal if the damping is set to a large value and then the Levenberg – Marquardt update step is near the steepest descent direction. On contrary, if the damping factor λ is too small, the Levenberg – Marquardt step approaches the exact quadratic step like the Gauss-Newton problem. Accordingly, Levenberg – Marquardt increases or decreases the damping value if a step fails to reduce the objective function. On this basis, Levenberg – Marquardt is capable to switch between a slow descent approach when being far from the minimum and a fast approach when being close to the minimum (Fig.1).
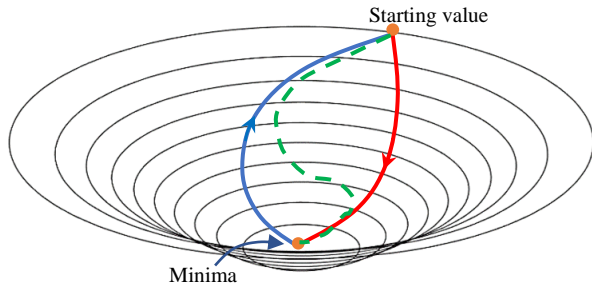

Fig.1. Levenberg-Marquardt method (dashed line) starts like gradient descent (blue) and switch to Gauss-Newton (red) when become in the neighborhood of the minima.

Accordingly, in this paper, we rely on two aspects to apply a vigorous 3D resection solution: 1) robust angular based mathematical model and 2) robust NLLS solution using the Levenberg – Marquardt method.

In the following section 2, the proposed method will be described in detail. In section 3, two experimental tests will be demonstrated to investigate the efficiency of the proposed method, and then discussion and conclusions will be shown in section 4.

II. METHOD

As mentioned, to ensure a vigorous solution of the proposed 3D resection problem, two models are followed:

- Robust geometric relationship between the observed angles with high redundant observations.
- Efficient least squares method that can vigorously handle the nonlinear redundant model.

Accordingly, a mathematical model is developed based on the oblique angles enclosed between the unknown point of resection and the reference points. Furthermore, two least-squares solutions are investigated to see how they perform when very far starting values are given and with the existence of redundant observations in a nonlinear model.

According to [6], oblique angles can be derived using directly measured two vertical angles (β_1, β_2) and one horizontal angle θ using the cosine rule of a spherical triangle as in equation 1:

$$\cos \gamma = \cos \theta \cos \beta_1 \cos \beta_2 + sin\beta_1 \, sin\beta_2 \qquad (1)$$

Furthermore, oblique angles can be defined using three points P, i, and j as shown in Fig. 2 where P is the unknown point. Consequently, a minimum of three oblique angles is necessary to define the 3D coordinates XYZ of a point resected by three rays. The maximum number of derived (observed) oblique angles can be calculated based on the number of the observed reference points n using equation 2:

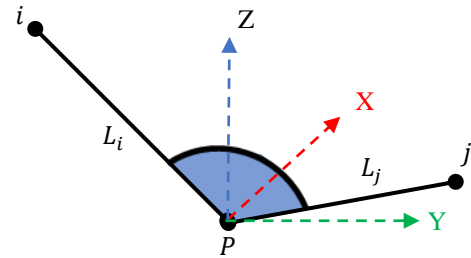$$Max. \, no. \, of \, oblique \, angles = \frac{(n^2 - n)}{2} \qquad (2)$$

Fig.2. One oblique angle enclosed between the reference points i, j and the unknown point P.

Differently, the computed oblique angles are based on the approximate unknown coordinates at the resection point as in equation 3:

$$cos\gamma = \frac{dx_i dx_j + dy_i dy_j + dz_i dz_j}{L_i * L_j} \qquad (3)$$

Where

$dx_i, dy_i, dz_i, dx_j, dy_j, dz_j$: the difference in coordinates between the unknown point P and the reference points i and j.
L_i, L_j : the spatial distance between the unknown point and the observed reference points i and j respectively.

To apply the nonlinear least squares solution with the existence of redundancy, the first-order derivative terms $(\frac{\partial F}{\partial X_P}, \frac{\partial F}{\partial Y_P}, \frac{\partial F}{\partial Z_P})$ of a Tylor expansion are evaluated as follows [13]:

$$F = L_i L_j \cos\gamma - (dx_i dx_j + dy_i dy_j + dz_i dz_j) \tag{4}$$

$$\frac{\partial F}{\partial X_P} = X_i - 2X_P + X_j + \frac{\cos\gamma \ (2X_P - 2X_i)L_j}{2L_j} + \frac{\cos\gamma \ (2X_P - 2X_j)L_i}{2L_j}$$

$$\frac{\partial F}{\partial Y_P} = Y_i - 2Y_P + Y_j + \frac{\cos\gamma \ (2Y_P - 2Y_i)L_j}{2L_i} + \frac{\cos\gamma \ (2Y_P - 2Y_j)L_i}{2L_j} \tag{5}$$

$$\frac{\partial F}{\partial Z_P} = Z_i - 2Z_P + Z_j + \frac{\cos\gamma \ (2Z_P - 2Z_i)L_j}{2L_i} + \frac{\cos\gamma \ (2Z_P - 2Z_j)L_i}{2L_j}$$

Where (X_P, Y_P, Z_P) represent the coordinates of the unknown point of resection.

Regardless of the method of a solution either using Gauss-Newton or Levenberg – Marquardt, the least-squares principle is based on minimizing the squared residual errors $v^t v = minimum$. Accordingly, the corrections Δ are iteratively computed using the Gauss-Newton method as:

$$\Delta = (B^t W B)^{-1} B^t W F \tag{6}$$

Where

B : Matrix of partial derivatives to the unknown coordinates X_P, Y_P, Z_P, $B = \begin{bmatrix} \frac{\partial F}{\partial X_P} & \frac{\partial F}{\partial Y_P} & \frac{\partial F}{\partial Z_P} \end{bmatrix}$.

W: Weight matrix of the observations.

The NLLS method using Gauss-Newton can be applied using the following steps:

1- Start with initial unknown parameters X^i and the mesured oblique angles.
2- Set up the stopping criteria like with a maximum number of iterations or when corrections Δ reach a very small tolerance threshold.
3- Compute the observation matrices of B^i and F^i based on the initial unknown parameters X^i using equation 4 and equation 5.
4- Compute the normal equation matrices using equation 6 and then compute corrections Δ^{i+1}:
5- update the unknown coordinates $X^{i+1} = X^i + \Delta^{i+1}$
6- Repeat step 3 and compute B^{i+1} and F^{i+1} using X^{i+1}
7- Check the condition of step 2:
   if $\|\Delta^{i+1}\| < 0.00001$ **or** no. of iterations > threshold
   stop
   else
   Repeat iterations to compute Δ^{i+2}.
8- Continue the same procedure until the conditions of step 7 are fulfilled.

On the other hand, the observation model using NLLS Levenberg – Marquardt method can be formulated using the sum of the *Misfit* which is mathematically expressed as the $v^t v$ and the *Model Norm* which is mathematically expressed as $\Delta^t \Delta$. Therefore, the objective function $\emptyset$ of the Levenberg – Marquardt least-squares method is to minimize $\emptyset$ after adding the effect of the damping parameter λ . As a result, the corrections Δ using Levenberg – Marquardt are computed as follows [12]:

$$\Delta = (B^t W B + \lambda I)^{-1} (B^t W F) \tag{7}$$

The benefit of adding the damping factor λ of equation 7 is to reduce the minimum norm solution. However, attention should be given since larger values of λ lead to underfitting while small values of λ lead to data overfitting. Hence, the least-squares method can be applied using the following steps [12]:

1- Start with a value of the damping factor like $\lambda = 1$ and initial unknown parameters X^i.
2- Determine the stopping criteria like with a maximum number of iterations of 50 with a very mall tolerance threshold.
3- Compute the observation matrices of B^i and F^i based on the initial unknown parameters X^i.
4- Compute the normal equation matrices with the addition of λ^i to the main diagonal of B^i . Then compute corrections Δ^{i+1}:
   Checking the stability of the solution and convergence through the following steps:
   if
   $$\frac{\|\Delta^{i+1}\|}{\|\Delta^i\| + tolerance} < tolerance$$
   then iterations **stop**
   else
   update the unknowns $X^{i+1} = X^i + \Delta^{i+1}$
5- Repeat step 3 and compute B^{i+1} and F^{i+1} using X^{i+1}
6- Check the condition:
   if $\|F^{i+1}\| < \|F^i\|$
   $\lambda^{i+1} = \sqrt{\lambda^i}$
   else
   $\lambda^{i+1} = 2\lambda^i$
7- Repeat iterations and start at step 4 to compute Δ^{i+2}.
8- Continue the same procedure until the conditions are fulfilled.

Subsequently, the covariance matrix Σ of the unknown point coordinates is computed in both mentioned solution methods as shown in equation 8.

$$\Sigma = \sigma_o^2 (B^t W B)^{-1} \tag{8}$$

where σ_o^2 is the variance of unit weight.

III. EXPERIMENTAL TEST

Two experimental tests are applied to investigate the efficiency of the proposed 3D resection method. The first experiment is based on simulation data while the second test is applied to real-world data.

A. Simulated test

A simulated resection problem is designed where the true position is known as $(10, -5, 2)$ and measured from four reference points as shown in Table I.

TABLE 1. FOUR SIMULATED REFERENCE POINTS

Point	X [m]	Y [m]	Z [m]
A	6.924	-14.972	0.000
B	5.813	-7.945	19.884
C	54.687	45.363	18.271
D	29.085	-76.020	21.878

The horizontal and vertical angles are observed and then the oblique angles are derived between the unknown point P and the reference points as illustrated in Table II using equation 1.

TABLE II. DERIVED OBLIQUE ANGLES OF THE SIMULATION TEST.

Angle	Oblique γ [Deg.]	Horizontal θ [Deg.]	Vertical β_1 [Deg.]	Vertical β_2 [Deg.]
88.218	38.175	-10.875	74.025	88.218
155.599	155.175	-10.875	13.575	155.599
40.848	31.725	-10.875	15.225	40.848
91.985	166.650	74.025	13.575	91.985
69.896	69.900	74.025	15.225	69.896
117.088	123.450	13.575	15.225	117.088

Starting values of the unknown point coordinates are selected as (0,0,0) to verify the robustness of the solution. The proposed NLLS adjustment using Gauss-Newton and Levenberg – Marquardt are applied and the correction iterations converged to zero as shown in Fig. 3.
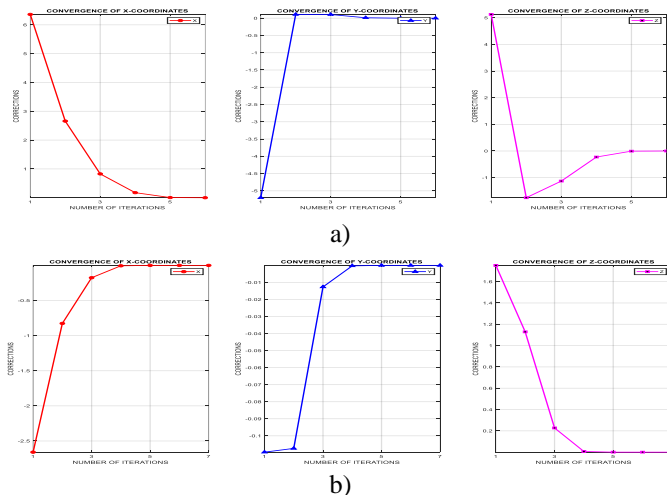

Fig.3. Corrections Δ of the simulated 3D resection point when starting @ (0,0,0) using a) Gauss-Newton method. b) Levenberg – Marquardt method.

The results of using the Gauss-Newton and Levenberg – Marquardt methods of the simulated test using starting values (0,0,0) are both identical as Xp=10.029 ± 0.066 m, Yp=-4.970 ± 0.034 m and Zp=2.001± 0.100 m.

However, if we select far starting values of $(10^{10}, 10^{10}, 10^{10})$ then the solution using Gauss-Newton will fail. Though the Levenberg – Marquardt method can steadily converge and corrections reach negligible values after 37 iterations to the global minimum of the true coordinates values as shown in Fig. 4. The adjusted coordinates are similar to the computed results when using starting values (0,0,0) as Xp=10.029 ± 0.066, Yp=-4.970 ± 0.034, and Zp=2.001 ± 0.100. It should be noted that a damping factor $\lambda = 1$ is selected when using the Levenberg – Marquardt method.
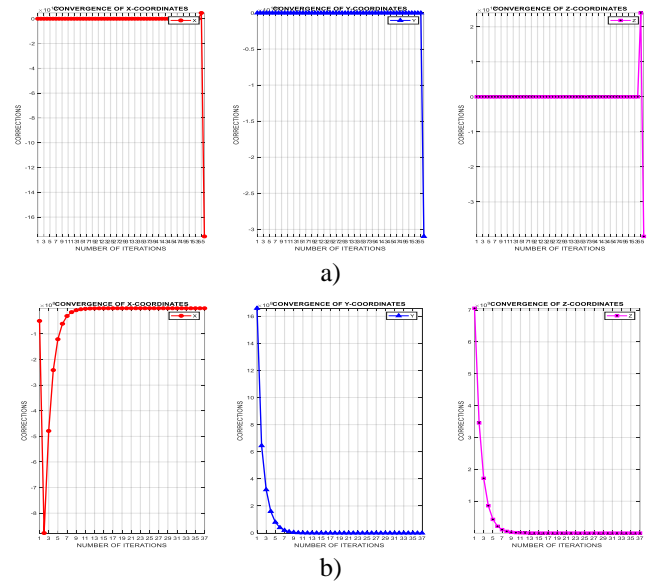

Fig. 4. Corrections Δ of the simulated 3D resection point when starting @ $(10^{10}, 10^{10}, 10^{10})$ using a) Gauss-Newton method. b) Levenberg – Marquardt method.

Fig.5 illustrates the gradual convergence of the estimated resection point position for every iteration until reaching the correct coordinates.
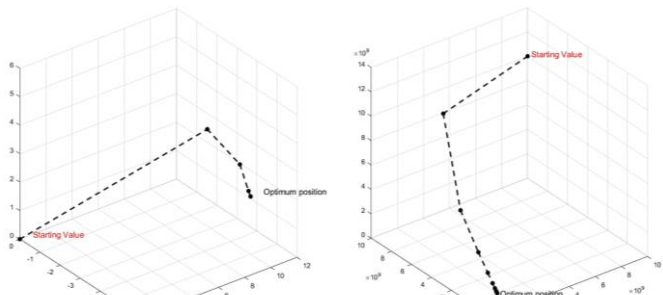

Fig.5. The adjusted simulated resection point converges to the true position during the iterative NLLS Levenberg – Marquardt method. Left) starting @ (0,0,0). Right) starting @ $(10^{10}, 10^{10}, 10^{10})$

B. Real-world test

Given five reference points measured to the nearest millimeter as shown in Table III.

TABLE III. FIVE REFERENCE POINTS.

Point	X [m]	Y [m]	Z [m]
A	92291.345	437615.913	38.831
B	92295.440	437603.790	19.308
C	92268.879	437625.464	43.961
D	92276.070	437610.980	-0.352
E	92243.252	437618.202	53.091

The horizontal and vertical angles are observed and then the oblique angles are derived between the unknown point P and the reference points as illustrated in Table IV using equation 1 and the angles are given to the fifth decimal place for demonstration.

TABLE IV. DERIVED OBLIQUE ANGLES.

Angle	Oblique γ [Deg.]	Horizontal θ [Deg.]	Vertical β_1 [Deg.]	Vertical β_2 [Deg.]
APB	24.80589	18.85500	43.46700	24.07875
APC	27.74341	39.60750	43.46700	54.20175
APD	48.10406	6.66000	43.46700	-4.26000
APE	54.43767	97.76250	43.46700	64.92975
BPC	52.39275	58.46250	24.07875	54.20175
BPD	37.68708	25.51500	24.07875	-4.26000
BPE	78.68345	116.61750	24.07875	64.92975
CPD	64.57988	32.94750	54.20175	-4.26000
CPE	30.06680	58.15500	54.20175	64.92975
EPA	94.32506	91.10250	-4.26000	64.92975

To test the strength of the suggested algorithm, challenging starting values of (0,0,0) for the unknown point coordinates are selected. The least-squares solution of the resection point using Gauss newton converged to the optimal values and corrections reaches negligible values after 22 iterations as shown in Fig. 6a. When using the Levenberg – Marquardt, the solution converged to global optimal, and corrections reach negligible values after 19 iterations using a damping factor $\lambda = 1$ as shown in Fig. 6b. It should be noted that both methods converged steadily to the optimal values at the 10th iteration of the adjusted three unknown coordinates.

The final coordinates of the resection points are 92255.797 m, 437597.078 m, and 2.647 m. However, if a very far starting value has been selected like (-10^8, -10^8, -10^8), then the Gauss-Newton method fails to converge as shown in Fig. 7a and end up to Nan values. Whilst the Levenberg – Marquardt method converges to the optimal values steadily and got close to the neighborhood of the minima at the 10th iteration and reaches a negligible correction after the 31st iteration as shown in Fig. 7b and Table V.

Fig.7. Corrections Δ of the 3D resection point when starting at (-10^8, -10^8, -10^8) using a) Gauss-Newton method. b) Levenberg – Marquardt method.

Fig. 8 shows the improvement of the adjusted point position in every iteration until reaches the optimum position.

Fig 8. The adjusted resection point converges to the correct position during the iterative NLLS Levenberg – Marquardt method when starting at (-10^8, -10^8, -10^8).
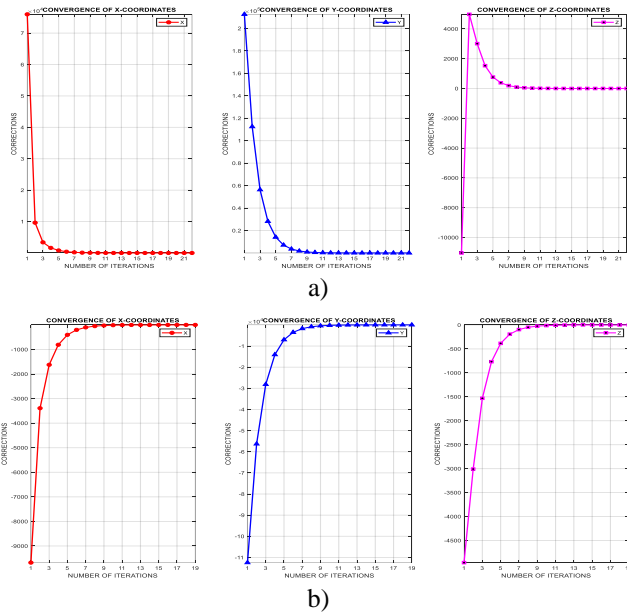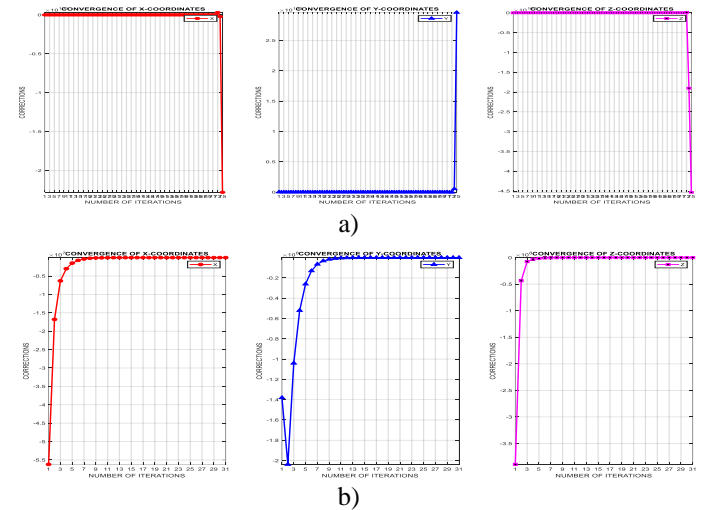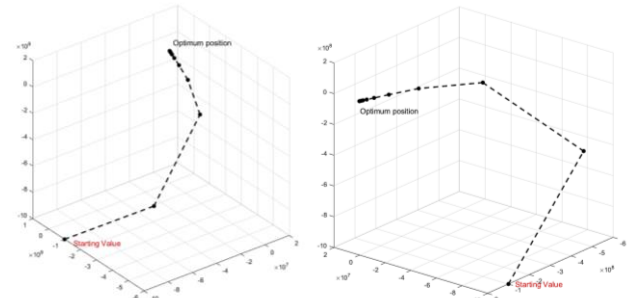

Fig 6. Corrections Δ of the 3D resection point when starting at (0,0,0) using a) Gauss-Newton method. b) Levenberg – Marquardt method.

TABLE V. THE 3D COORDINATES OF THE RESECTION POINT USING LEVENBERG – MARQUARDT AND GAUSS-NEWTON WHEN STARTING FROM (-10^8, -10^8, -10^8).

		X [m]	Y [m]	Z [m]
Correct	LM method	92255.797	437597.078	2.647
Failed	GN method	Nan	Nan	Nan

IV. DISCUSSION AND CONCLUSIONS

In this paper, a vigorous 3D resection approach is proposed relying on two aspects: 1) a robust geometric model of oblique angles and 2) an efficient nonlinear least-squares method using the Levenberg – Marquardt algorithm.

Two experiments are applied using simulated and real-world data sets. Both least-squares techniques using Gauss-Newton and Levenberg – Marquardt are applied.

For both tests, the Gauss-Newton method was stable and converges to the global minimum when starting at (0,0,0) values which indicated a good performance as shown in Fig.3a and Fig.6a. However, Levenberg – Marquardt outperformed the Gauss-Newton method in both tests when started at very challenging values of the unknown coordinates and steadily converged to the global minimum as shown in Fig.4b and Fig.7b. It should be noted that in the simulated test, the results did not exactly match the true values because of the deliberate approximation of the measured angles to the 3rd decimal place of a degree as shown in Table II.

As a final conclusion, the proposed method of the angular based 3D resection proved its efficiency and stability against improper starting values, especially when using the nonlinear least-square adjustment powered by the Levenberg – Marquardt algorithm.

Future work will continue expanding the 3D resection model and combining it with an intersection model to have a complete vigorous positioning algorithm that can be used in solving different challenging problems in surveying and geospatial engineering.

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