Vol. 06, No. 02, pp. 149 –160 (2025)
ISSN: 2708-0757

JOURNAL OF APPLIED SCIENCE AND TECHNOLOGY TRENDS

www.jastt.org

Enhanced algorithm based on Chio-like Method for Non-Square Determinant Calculations for application in CBVR

Besnik Duriqi[1,*], Halil Snopçe[1], Armend Salihu[2], Artan Luma[3], Majlinda Fetaji[4]

[1]*Computer Science, South East European University (SEEU), North Macedonia, bd30521@seeu.edu.mk*, h.snopce@seeu.edu.mk
[2]*Computer Science, UNI Universum International College, Kosovo, ar.salihu@gmail.com*
[3]*Computer Science, South East European University (SEEU), North Macedonia, a.luma@seeu.edu.mk*
[4]*Computer Science, South East European University (SEEU), North Macedonia,* m.fetaji@seeu.edu.*mk*

**Correspondence: bd30521@seeu.edu.mk*

Abstract

In this paper, we propose an enhanced algorithm based on the Chio-like method for calculating non-square determinants, optimized for content-based video retrieval (CBVR) systems. The algorithm accelerates the computation of determinant kernels used for similarity score generation, which is critical for efficient video indexing and retrieval. While the classical Chio-like method reduces the determinant order by one at each step, our improved approach reduces the order by four, providing notable computational benefits. Although the asymptotic time complexity remains the same, considering the fact that the resulting determinant is decreased by four orders compared to one and two orders, respectively, from existing Chio-like methods, the proposed method demonstrates clear practical performance improvements. The computer implementation of the proposed algorithm in MATLAB shows an average execution time reduction of approximately 24.5% compared to the standard Chio-like method and 3.2% compared to its modified version. These enhancements make the method well-suited for large-scale or real-time CBVR applications, where fast and accurate similarity evaluation is essential.

Keywords: *Non-square determinants, Chio's-like, Algorithm optimization, Execution time, Kernels, Similarity score, CBVR*

Received: June 06th, 2025 / Revised: July 17th, 2025 / Accepted: July 30th, 2025 / Online: August 01st, 2025

I. INTRODUCTION

This paper introduces an optimized Chio-like algorithm specifically designed to enhance the performance of determinant kernels in Content-based video retrieval (CBVR) systems. By reducing the order of determinants more aggressively and optimizing computational steps, our method enables faster determinant kernels calculations that are essential for similarity score generation, thus improving the overall responsiveness and scalability of CBVR platforms. The remainder of this paper outlines the mathematical formulation, algorithm design, complexity analysis, and simulation results validating the proposed approach.

In the context of content-based video retrieval (CBVR), fast and accurate similarity computation is essential for identifying relevant video segments [1]. Determinant-based kernels have emerged as a powerful alternative to traditional similarity measures, such as cosine similarity, due to their ability to capture higher-order structural relationships [2-4]. Kernel functions belong to the branch of mathematics known as operator theory [5 - 7] and they have shown as a very efficient approach in image and video retrieval, machine learning etc. [8-9]. Kernel functions operate on vectors; square and non-square matrices and they define the inner product between the shot-feature matrices and yield a real number as the result. A matrix on the other hand is an array of numbers or symbols organized in rows and columns. The kernel of a matrix is the collection of vectors that, when multiplied by the matrix, produce a zero vector. A matrix's determinant is a scalar value calculated from its members that represents certain matrix features. In our research aim, they provide a robust way for similarity measure in retrieval systems and are defined as inner product between feature matrices by returning a real number as a result which indicates the degree of similarity in between query video versus database videos [10 - 11]. Because the feature matrices of real multimedia are non-square, where the number of columns and rows are not

doi:10.38094/jastt62253

equal, hence, it requires to address those as such in order to avoid missing the full data due to summarization and in this way, it requires processing the non-square determinant calculation for similarity score generation as fast and efficient as possible by preserving the quality of the result [12 - 13]. The efficiency of these kernels directly depends on the underlying determinant calculation algorithms.

Non-square determinant theory has historically been developed to extend classical square matrix determinant concepts to non-square matrices. The initial framework was introduced by Cullis in 1913 in *Matrices and Determinandoids* [14], and further formalized by Radic in 1966 [15], as presented in the following formula:

$$det(A_{m\times n}) = |A_{m\times n}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}$$

$$= \sum_{1<j_1<\cdots<j_m<n} (-1)^{r+s} \begin{vmatrix} a_{1j_1} & a_{1j_2} & \cdots & a_{1j_m} \\ a_{2j_1} & a_{2j_2} & \cdots & a_{2j_m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{mj_1} & a_{mj_2} & \cdots & a_{mj_m} \end{vmatrix} \quad (1)$$

Where $r = 1 + \cdots + m, s = j_1 + \cdots + j_m$.

The algorithmic approach of Radic definition is presented on Algorithm 1 as following:

Algorithm 1: det_Radic algorithm based on Radic method for non-square determinant calculations

```
Step 1: Insert order of non-square
        determinant of order mxn
Step 2: Insert non-square determinant A
Step 3: Calculate non-square determinants
        using Radic Method
      Initialize d = 0;
      Identify all combinations for
      determining mxn square determinants
      Create Loop for i from 1 to total
      number   of   combinations   d=d+(-
1)^(sum(1:m)+sum(B(i,1:m)))*SquareDet((A(
1:m,B(i,1:m))));
      end
Step 4: Display the result of the
        Determinant
```

Another approach is developed by [16], which is based on classical Chio condensation method that is implemented to non-square matrices as it is presented in Theorem 1. Further [17] improve this approach to reduce the order of non-square determinant for two orders as it is presented in Theorem 2. The above-mentioned theorems are the foundation and motivation for the new approach presented in this paper to reduce the order of determiant for four orders.

Theorem 1 [16]: (Chio's-like method for non-square determinants): For a non-square determinant of order $m \times n$, in cases for 2×3, 2×4 and 3×4, the following formula holds:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}_{m\times n} \quad (2)$$

$$= \frac{|A_c|}{a_{11}^{m-2}} (-1)^m \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1n} \\ a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2} & a_{m3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-1)},$$

Where:

$$|A_c| = \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{m1} & a_{m2} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{m1} & a_{mn} \end{vmatrix} \end{vmatrix}_{(m-1)\times(n-1)} \quad (3)$$

and $a_{11} \neq 0$.

Proof: see [16]

Theorem 2[17]: Suppose that A is non-square matrix of order $m \times n, m > 3$ and $m \leq n - 1$, its determinant can be calculated using formula below:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}_{m\times n} = \frac{|A_{c1}|}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}^{m-3}} \quad (4)$$

$$+(-1)^m \begin{vmatrix} a_{12} - a_{11} & a_{13} & \cdots & a_{1n} \\ a_{22} - a_{21} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2} - a_{m2} & a_{m3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-1)}$$

Where:

$$|A_{c1}| = \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{31} & a_{32} & a_{3n} \end{vmatrix} \\ \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{m1} & a_{m2} & a_{m3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{m1} & a_{m2} & a_{mn} \end{vmatrix} \end{vmatrix}_{(m-2)\times(n-2)}$$

(5)

and $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0.$

Proof: see [17]

In the following are presented also algorithmic approach of Theorem 1 and Theorem 2.

Algorithm 2 is based on Theorem 1 as following:

Algorithm 2: Recursive algorithm det_Chio for Theorem 1 (Chio's-like) method to calculate non-square determinants of order $m \times n$

```
Step 1: Insert the non-square determinant A
Step 2: Determine the order of non-square determinant m×n
      [m,n] = size(A);
Step 3: Checking if A(1,1) is equal to 0
      if A(1,1) = 0
            Exchange rows to find nonzero element
Step 4: Calculating sub matrices
      Initialize B=0;
      Create Loop for i from 1 to m-1
            Create Loop for j from 1 to n-1
                  B(i,j) = A(1,1) * A(i + 1, j + 1) - A(1, j + 1) * A(i + 1, 1)
            end
      end
Step 5: Calculate the final result of non-square determinant
      d = 1/A(1,1)^(m - 2) * det_Chio(B) + (-1)^m * det_Chio(A(1:m, 2:n));
Step 5: Display the result of the determinant
```

Algorithm 3 is based on Theorem 2 as following:

Algorithm 3: det_Chio2 algorithm for Theorem 2 (Modified Chio's-like) method to calculate non-square determinants of order $m \times n$

```
Step 1: Checking for conditions:
      if m>n
            d=0;
            return
      end
      if m==n
            Calculate determinant with Radic Definition
            return
      end
      if m<3
            Calculate determinant with Radic Definition
            return
      end
Step 2: Calculate pivot block:
      Pivot = det(A(1:2,1:2));
      if Pivot == 0
            Interchange rows to have another nonzero pivot block
      end
Step 3: Calculating sub matrices
      Initialize B=zeros(m-2,n-2);
      Create Loop for i from 3 to m
            Create Loop for j from 3 to n
                  B(i-2, j-2) = det(A([1 2 i], [1 2 j]));
            end
      end
Step 4: Calculate the result of non-square determinant
      Sub1 = A(:,2) - A(:,1) //m×1
      d=det_Chio2(B)/Pivot^(m-3)+(1)^m*det_Chio2([Sub1 A(:,3:n)]);
```

(16)

In section 2 is presented the new theorem that is enhanced Chio-like method based on previous research and it continues with section 3 where its algorithmic approach is presented as well as simulation results, and those two sections provide the foundation of the research. Then we continue with section 4, where the conclusion of the research is presented.

II. Enhanced Methodology For Calculating Non-Square Determinant

In order to prove the main results, we need the following Lemmas:

Lemma 1: For the fifth order block determinant the following formula holds:

$$\left\| \begin{matrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{15} \\ a_{21} & a_{22} & a_{25} \\ a_{31} & a_{32} & a_{35} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{15} \\ a_{21} & a_{22} & a_{25} \\ a_{41} & a_{42} & a_{45} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{51} & a_{52} & a_{53} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{51} & a_{52} & a_{54} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{15} \\ a_{21} & a_{22} & a_{25} \\ a_{51} & a_{52} & a_{55} \end{vmatrix} \end{matrix} \right\| =$$

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}^2 \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} \qquad (6)$$

Proof see [18].

Lemma 2: $det(B) = k \cdot det(A)$ if a non-square matrix B is produced from multiplying elements of a row of a non-square matrix A with a scalar k [19-20].

Lemma 3: For the fourth order block determinant the following formula holds:

$$\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \tag{7}$$

Proof see [18].

Theorem 3: Suppose that A is non-square matrix of order $m \times n, m > 5$ and $m \leq n-1$, its determinant can be calculated using formula below:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}_{m\times n} = \frac{|A_{c2}|}{\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}^{m-5}}$$

$$+ \begin{vmatrix} a_{11} & a_{12} & a_{14}-a_{13} & a_{15} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{24}-a_{23} & a_{25} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m4}-a_{m3} & a_{m5} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-1)}$$

$$+ \begin{vmatrix} a_{12}-a_{11} & a_{14}-a_{13} & \cdots & a_{1n} \\ a_{22}-a_{21} & a_{24}-a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2}-a_{m1} & a_{m4}-a_{m3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-2)} \tag{8}$$

$$+(-1)^m \begin{vmatrix} a_{12}-a_{11} & a_{13} & \cdots & a_{1n} \\ a_{22}-a_{21} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2}-a_{m1} & a_{m3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-1)}$$

Where

$$|A_{c2}| = \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{3n} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{4n} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{5n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{3n} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{4n} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{6n} \end{vmatrix} \\ \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{m1} & a_{m2} & a_{m3} & a_{m4} & a_{m5} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{3n} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{4n} \\ a_{m1} & a_{m2} & a_{m3} & a_{m4} & a_{mn} \end{vmatrix} \end{vmatrix}_{(m-4)\times(n-4)} \tag{9}$$

Proof: Implementing recursively Theorem 2 in $|A_{c1}|$ (equation 5) we have:

$$|A_{c1}| = \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{31} & a_{32} & a_{3n} \end{vmatrix} \\ \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{m1} & a_{m2} & a_{m3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{m1} & a_{m2} & a_{mn} \end{vmatrix} \end{vmatrix}_{(m-2)x(n-2)} = \frac{|A_{c3}|}{\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} \end{vmatrix}_{(m-2)-3}} +$$

$$+(-1)^{(m-2)}\begin{vmatrix}
\begin{vmatrix} a_{11} & a_{12} & a_{14}\\ a_{21} & a_{22} & a_{24}\\ a_{31} & a_{32} & a_{34}\end{vmatrix}-\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{15}\\ a_{21} & a_{22} & a_{25}\\ a_{31} & a_{32} & a_{35}\end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{1n}\\ a_{21} & a_{22} & a_{2n}\\ a_{31} & a_{32} & a_{3n}\end{vmatrix}\\
\begin{vmatrix} a_{11} & a_{12} & a_{14}\\ a_{21} & a_{22} & a_{24}\\ a_{41} & a_{42} & a_{44}\end{vmatrix}-\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{41} & a_{42} & a_{43}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{15}\\ a_{21} & a_{22} & a_{25}\\ a_{41} & a_{42} & a_{45}\end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{1n}\\ a_{21} & a_{22} & a_{2n}\\ a_{41} & a_{42} & a_{4n}\end{vmatrix}\\
\vdots & \vdots & \ddots & \vdots\\
\begin{vmatrix} a_{11} & a_{12} & a_{14}\\ a_{21} & a_{22} & a_{24}\\ a_{m1} & a_{m2} & a_{m4}\end{vmatrix}-\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{m1} & a_{m2} & a_{m3}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{15}\\ a_{21} & a_{22} & a_{25}\\ a_{m1} & a_{m2} & a_{m5}\end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{1n}\\ a_{21} & a_{22} & a_{2n}\\ a_{m1} & a_{m2} & a_{mn}\end{vmatrix}
\end{vmatrix}_{(m-2)\times((n-2)-1)} \tag{10}$$

Where:

$|A_{c3}| =$

$$\begin{vmatrix}
\begin{vmatrix}
\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14}\\ a_{21} & a_{22} & a_{24}\\ a_{31} & a_{32} & a_{34}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{15}\\ a_{21} & a_{22} & a_{25}\\ a_{31} & a_{32} & a_{35}\end{vmatrix}\\
\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{41} & a_{42} & a_{43}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14}\\ a_{21} & a_{22} & a_{24}\\ a_{41} & a_{42} & a_{44}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{15}\\ a_{21} & a_{22} & a_{25}\\ a_{41} & a_{42} & a_{45}\end{vmatrix}\\
\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{51} & a_{52} & a_{53}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14}\\ a_{21} & a_{22} & a_{24}\\ a_{51} & a_{52} & a_{54}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{15}\\ a_{21} & a_{22} & a_{25}\\ a_{51} & a_{52} & a_{55}\end{vmatrix}
\end{vmatrix} & \cdots & \begin{vmatrix}
\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14}\\ a_{21} & a_{22} & a_{24}\\ a_{31} & a_{32} & a_{34}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{1n}\\ a_{21} & a_{22} & a_{2n}\\ a_{31} & a_{32} & a_{3n}\end{vmatrix}\\
\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{41} & a_{42} & a_{43}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14}\\ a_{21} & a_{22} & a_{24}\\ a_{41} & a_{42} & a_{44}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{1n}\\ a_{21} & a_{22} & a_{2n}\\ a_{41} & a_{42} & a_{4n}\end{vmatrix}\\
\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{51} & a_{52} & a_{53}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14}\\ a_{21} & a_{22} & a_{24}\\ a_{51} & a_{52} & a_{54}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{1n}\\ a_{21} & a_{22} & a_{2n}\\ a_{51} & a_{52} & a_{5n}\end{vmatrix}
\end{vmatrix}\\
\vdots & \ddots & \vdots\\
\begin{vmatrix}
\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14}\\ a_{21} & a_{22} & a_{24}\\ a_{31} & a_{32} & a_{34}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{15}\\ a_{21} & a_{22} & a_{25}\\ a_{31} & a_{32} & a_{35}\end{vmatrix}\\
\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{41} & a_{42} & a_{43}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14}\\ a_{21} & a_{22} & a_{24}\\ a_{41} & a_{42} & a_{44}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{15}\\ a_{21} & a_{22} & a_{25}\\ a_{41} & a_{42} & a_{45}\end{vmatrix}\\
\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{m1} & a_{m2} & a_{m3}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14}\\ a_{21} & a_{22} & a_{24}\\ a_{m1} & a_{m2} & a_{m4}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{15}\\ a_{21} & a_{22} & a_{25}\\ a_{m1} & a_{m2} & a_{m5}\end{vmatrix}
\end{vmatrix} & \cdots & \begin{vmatrix}
\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14}\\ a_{21} & a_{22} & a_{24}\\ a_{31} & a_{32} & a_{34}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{1n}\\ a_{21} & a_{22} & a_{2n}\\ a_{31} & a_{32} & a_{3n}\end{vmatrix}\\
\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{41} & a_{42} & a_{43}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14}\\ a_{21} & a_{22} & a_{24}\\ a_{41} & a_{42} & a_{44}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{1n}\\ a_{21} & a_{22} & a_{2n}\\ a_{41} & a_{42} & a_{4n}\end{vmatrix}\\
\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{m1} & a_{m2} & a_{m3}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14}\\ a_{21} & a_{22} & a_{24}\\ a_{m1} & a_{m2} & a_{m4}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{1n}\\ a_{21} & a_{22} & a_{2n}\\ a_{m1} & a_{m2} & a_{mn}\end{vmatrix}
\end{vmatrix}
\end{vmatrix}_{(m-4)\times(n-4)} \tag{11}$$

Based on Lemma 1 we have:

$$|A_{c3}| = \begin{vmatrix}
\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}^2 \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15}\\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25}\\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35}\\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45}\\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55}\end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}^2 \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{1n}\\ a_{21} & a_{22} & a_{23} & a_{24} & a_{2n}\\ a_{31} & a_{32} & a_{33} & a_{34} & a_{3n}\\ a_{41} & a_{42} & a_{43} & a_{44} & a_{4n}\\ a_{51} & a_{52} & a_{53} & a_{54} & a_{5n}\end{vmatrix}\\
\vdots & \ddots & \vdots\\
\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}^2 \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15}\\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25}\\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35}\\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45}\\ a_{m1} & a_{m2} & a_{m3} & a_{m4} & a_{m5}\end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}^2 \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{1n}\\ a_{21} & a_{22} & a_{23} & a_{24} & a_{2n}\\ a_{31} & a_{32} & a_{33} & a_{34} & a_{3n}\\ a_{41} & a_{42} & a_{43} & a_{44} & a_{4n}\\ a_{m1} & a_{m2} & a_{m3} & a_{m4} & a_{mn}\end{vmatrix}
\end{vmatrix}_{(m-4)\times(n-4)} \tag{12}$$

Implementing Lemma 2 in equation (12) we receive:

$$|A_{c3}|=\left(\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}^2\right)^{m-4}\cdot\left|\begin{matrix}\begin{vmatrix}a_{11}&a_{12}&a_{13}&a_{14}&a_{15}\\a_{21}&a_{22}&a_{23}&a_{24}&a_{25}\\a_{31}&a_{32}&a_{33}&a_{34}&a_{35}\\a_{41}&a_{42}&a_{43}&a_{44}&a_{45}\\a_{51}&a_{52}&a_{53}&a_{54}&a_{55}\end{vmatrix}&\cdots&\begin{vmatrix}a_{11}&a_{12}&a_{13}&a_{14}&a_{1n}\\a_{21}&a_{22}&a_{23}&a_{24}&a_{2n}\\a_{31}&a_{32}&a_{33}&a_{34}&a_{3n}\\a_{41}&a_{42}&a_{43}&a_{44}&a_{4n}\\a_{51}&a_{52}&a_{53}&a_{54}&a_{5n}\end{vmatrix}\\ \vdots&\ddots&\vdots\\ \begin{vmatrix}a_{11}&a_{12}&a_{13}&a_{14}&a_{15}\\a_{21}&a_{22}&a_{23}&a_{24}&a_{25}\\a_{31}&a_{32}&a_{33}&a_{34}&a_{35}\\a_{41}&a_{42}&a_{43}&a_{44}&a_{45}\\a_{m1}&a_{m2}&a_{m3}&a_{m4}&a_{m5}\end{vmatrix}&\cdots&\begin{vmatrix}a_{11}&a_{12}&a_{13}&a_{14}&a_{1n}\\a_{21}&a_{22}&a_{23}&a_{24}&a_{2n}\\a_{31}&a_{32}&a_{33}&a_{34}&a_{3n}\\a_{41}&a_{42}&a_{43}&a_{44}&a_{4n}\\a_{m1}&a_{m2}&a_{m3}&a_{m4}&a_{mn}\end{vmatrix}\end{matrix}\right|_{(m-4)\times(n-4)}$$

$$=\left(\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}^2\right)^{m-4}\cdot|A_{c2}| \tag{13}$$

Implementing equation 13 in equation 10 we have:

$$|A_{c1}|=\left|\begin{matrix}\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}&\cdots&\begin{vmatrix}a_{11}&a_{12}&a_{1n}\\a_{21}&a_{22}&a_{2n}\\a_{31}&a_{32}&a_{3n}\end{vmatrix}\\ \vdots&\ddots&\vdots\\ \begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{m1}&a_{m2}&a_{m3}\end{vmatrix}&\cdots&\begin{vmatrix}a_{11}&a_{12}&a_{1n}\\a_{21}&a_{22}&a_{2n}\\a_{m1}&a_{m2}&a_{mn}\end{vmatrix}\end{matrix}\right|_{(m-2)x(n-2)}=\frac{\left(\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}^2\right)^{m-4}\cdot|A_{c2}|}{\left|\begin{matrix}\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}&\begin{vmatrix}a_{11}&a_{12}&a_{14}\\a_{21}&a_{22}&a_{24}\\a_{31}&a_{32}&a_{34}\end{vmatrix}\\ \begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{41}&a_{42}&a_{43}\end{vmatrix}&\begin{vmatrix}a_{11}&a_{12}&a_{14}\\a_{21}&a_{22}&a_{24}\\a_{41}&a_{42}&a_{44}\end{vmatrix}\end{matrix}\right|^{(m-2)-3}}+ \tag{14}$$

$$+(-1)^{(m-2)}\left|\begin{matrix}\begin{vmatrix}a_{11}&a_{12}&a_{14}\\a_{21}&a_{22}&a_{24}\\a_{31}&a_{32}&a_{34}\end{vmatrix}-\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}&\begin{vmatrix}a_{11}&a_{12}&a_{15}\\a_{21}&a_{22}&a_{25}\\a_{31}&a_{32}&a_{35}\end{vmatrix}&\cdots&\begin{vmatrix}a_{11}&a_{12}&a_{1n}\\a_{21}&a_{22}&a_{2n}\\a_{31}&a_{32}&a_{3n}\end{vmatrix}\\ \begin{vmatrix}a_{11}&a_{12}&a_{14}\\a_{21}&a_{22}&a_{24}\\a_{41}&a_{42}&a_{44}\end{vmatrix}-\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{41}&a_{42}&a_{43}\end{vmatrix}&\begin{vmatrix}a_{11}&a_{12}&a_{15}\\a_{21}&a_{22}&a_{25}\\a_{41}&a_{42}&a_{45}\end{vmatrix}&\cdots&\begin{vmatrix}a_{11}&a_{12}&a_{1n}\\a_{21}&a_{22}&a_{2n}\\a_{41}&a_{42}&a_{4n}\end{vmatrix}\\ \vdots&\vdots&\ddots&\vdots\\ \begin{vmatrix}a_{11}&a_{12}&a_{14}\\a_{21}&a_{22}&a_{24}\\a_{m1}&a_{m2}&a_{m4}\end{vmatrix}-\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{m1}&a_{m2}&a_{m3}\end{vmatrix}&\begin{vmatrix}a_{11}&a_{12}&a_{15}\\a_{21}&a_{22}&a_{25}\\a_{m1}&a_{m2}&a_{m5}\end{vmatrix}&\cdots&\begin{vmatrix}a_{11}&a_{12}&a_{1n}\\a_{21}&a_{22}&a_{2n}\\a_{m1}&a_{m2}&a_{mn}\end{vmatrix}\end{matrix}\right|_{(m-2)\times((n-2)-1)}$$

Implementing Lemma 3 in equation 14 we have:

$$|A_{c1}|=\left|\begin{matrix}\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}&\cdots&\begin{vmatrix}a_{11}&a_{12}&a_{1n}\\a_{21}&a_{22}&a_{2n}\\a_{31}&a_{32}&a_{3n}\end{vmatrix}\\ \vdots&\ddots&\vdots\\ \begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{m1}&a_{m2}&a_{m3}\end{vmatrix}&\cdots&\begin{vmatrix}a_{11}&a_{12}&a_{1n}\\a_{21}&a_{22}&a_{2n}\\a_{m1}&a_{m2}&a_{mn}\end{vmatrix}\end{matrix}\right|_{(m-2)x(n-2)}=\frac{\left(\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}^2\right)^{m-4}\cdot|A_{c2}|}{\left(\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}\cdot\begin{vmatrix}a_{11}&a_{12}&a_{13}&a_{14}\\a_{21}&a_{22}&a_{23}&a_{24}\\a_{31}&a_{32}&a_{33}&a_{34}\\a_{41}&a_{42}&a_{43}&a_{44}\end{vmatrix}\right)^{(m-2)-3}}+ \tag{15}$$

$$+(-1)^{(m-2)}\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{15} \\ a_{21} & a_{22} & a_{25} \\ a_{31} & a_{32} & a_{35} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{31} & a_{32} & a_{3n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{15} \\ a_{21} & a_{22} & a_{25} \\ a_{41} & a_{42} & a_{45} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{41} & a_{42} & a_{4n} \end{vmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{m1} & a_{m2} & a_{m4} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{m1} & a_{m2} & a_{m3} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{15} \\ a_{21} & a_{22} & a_{25} \\ a_{m1} & a_{m2} & a_{m5} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{m1} & a_{m2} & a_{mn} \end{vmatrix} \end{vmatrix}_{(m-2)\times((n-2)-1)}$$

From Theorem 3 we have:

$$\begin{vmatrix} a_{11} & a_{12} & a_{14}-a_{13} & a_{15} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{24}-a_{23} & a_{25} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m2} & a_{m2} & a_{m4}-a_{m3} & a_{m5} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-1)} =$$

$$= \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{14}-a_{13} \\ a_{21} & a_{22} & a_{24}-a_{23} \\ a_{31} & a_{32} & a_{34}-a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{15} \\ a_{21} & a_{22} & a_{25} \\ a_{31} & a_{32} & a_{35} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{31} & a_{32} & a_{3n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} & a_{14}-a_{13} \\ a_{21} & a_{22} & a_{24}-a_{23} \\ a_{41} & a_{42} & a_{44}-a_{43} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{15} \\ a_{21} & a_{22} & a_{25} \\ a_{41} & a_{42} & a_{45} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{41} & a_{42} & a_{4n} \end{vmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} & a_{14}-a_{13} \\ a_{21} & a_{22} & a_{24}-a_{23} \\ a_{m1} & a_{m2} & a_{m4}-a_{m3} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{15} \\ a_{21} & a_{22} & a_{25} \\ a_{m1} & a_{m2} & a_{m5} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{m1} & a_{m2} & a_{mn} \end{vmatrix} \end{vmatrix}_{(m-2)\times((n-3))} - \qquad (16)$$

$$-(-1)^{m}\begin{vmatrix} a_{12}-a_{11} & a_{14}-a_{13} & \cdots & a_{1n} \\ a_{22}-a_{21} & a_{24}-a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2}-a_{m2} & a_{m4}-a_{m3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-2)}$$

After mathematical operations in equation 16, we have:

$$\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{14}-a_{13} \\ a_{21} & a_{22} & a_{24}-a_{23} \\ a_{31} & a_{32} & a_{34}-a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{15} \\ a_{21} & a_{22} & a_{25} \\ a_{31} & a_{32} & a_{35} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{1\mathrm{n}} \\ a_{21} & a_{22} & a_{2\mathrm{n}} \\ a_{31} & a_{32} & a_{3\mathrm{n}} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} & a_{14}-a_{13} \\ a_{21} & a_{22} & a_{24}-a_{23} \\ a_{41} & a_{42} & a_{44}-a_{43} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{15} \\ a_{21} & a_{22} & a_{25} \\ a_{41} & a_{42} & a_{45} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{1\mathrm{n}} \\ a_{21} & a_{22} & a_{2\mathrm{n}} \\ a_{41} & a_{42} & a_{4\mathrm{n}} \end{vmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} & a_{14}-a_{13} \\ a_{21} & a_{22} & a_{24}-a_{23} \\ a_{\mathrm{m}1} & a_{\mathrm{m}2} & a_{\mathrm{m}4}-a_{\mathrm{m}3} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{15} \\ a_{21} & a_{22} & a_{25} \\ a_{\mathrm{m}1} & a_{\mathrm{m}2} & a_{\mathrm{m}5} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{1\mathrm{n}} \\ a_{21} & a_{22} & a_{2\mathrm{n}} \\ a_{\mathrm{m}1} & a_{\mathrm{m}2} & a_{\mathrm{mn}} \end{vmatrix} \end{vmatrix}_{(m-2)\times((\mathrm{n}-3))} \tag{17}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{14}-a_{13} & a_{15} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{24}-a_{23} & a_{25} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{\mathrm{m}2} & a_{m2} & a_{\mathrm{m}4}-a_{\mathrm{m}3} & a_{\mathrm{m}5} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-1)} + (-1)^m \begin{vmatrix} a_{12}-a_{11} & a_{14}-a_{13} & \cdots & a_{1n} \\ a_{22}-a_{21} & a_{24}-a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2}-a_{\mathrm{m}2} & a_{\mathrm{m}4}-a_{\mathrm{m}3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-2)}$$

Implementing equation 17 in equation 15 we receive:

$$|A_{c1}| = \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{31} & a_{32} & a_{3n} \end{vmatrix} \\ \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{m1} & a_{m2} & a_{m3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{m1} & a_{m2} & a_{mn} \end{vmatrix} \end{vmatrix}_{(m-2)x(n-2)} = \frac{\left(\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}^2\right)^{m-4} \cdot |A_{c2}|}{\left(\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}\right)^{(m-2)-3}} + \tag{18}$$

$$+(-1)^{(m-2)}\left(\begin{vmatrix} a_{11} & a_{12} & a_{14}-a_{13} & a_{15} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{24}-a_{23} & a_{25} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{\mathrm{m}2} & a_{m2} & a_{\mathrm{m}4}-a_{\mathrm{m}3} & a_{\mathrm{m}5} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-1)} + (-1)^m \begin{vmatrix} a_{12}-a_{11} & a_{14}-a_{13} & \cdots & a_{1n} \\ a_{22}-a_{21} & a_{24}-a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2}-a_{\mathrm{m}2} & a_{\mathrm{m}4}-a_{\mathrm{m}3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-2)}\right)$$

Implementing equation 18 in equation 4 and after mathematical operations, we have:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}_{m\times n} = \frac{|A_{c2}|}{\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}^{m-5}} + \begin{vmatrix} a_{11} & a_{12} & a_{14}-a_{13} & a_{15} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{24}-a_{23} & a_{25} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m4}-a_{m3} & a_{m5} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-1)} \tag{19}$$

$$+ \begin{vmatrix} a_{12}-a_{11} & a_{14}-a_{13} & \cdots & a_{1n} \\ a_{22}-a_{21} & a_{24}-a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2}-a_{m1} & a_{m4}-a_{m3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-2)} + (-1)^m \begin{vmatrix} a_{12}-a_{11} & a_{13} & \cdots & a_{1n} \\ a_{22}-a_{21} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2}-a_{m1} & a_{m3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-1)}$$

Hence it is proved that Theorem 3 holds.

III. Algorithmic Approach For Non-Square Determinant Calculation

In this chapter, we proceed into developing the algorithmic approache for calculating non-square determinants based on the Theorem 3. The pseudocode of algorithm for non-square determinants calculation based on Theorem 3 is presented below:

Algorithm 4: det_Chio4 algorithm for Theorem 3 (Enhanced Chio's-like) method to calculate non-square determinants of order $\boldsymbol{m \times n, m > 5}$ and $\boldsymbol{m \leq n-1}$

```
Step 1: Check edge cases
      if m > n
            d = 0
            return d
      end
      if m == n or m < 5
            d = Radic_Definition(A) //
use Radic or standard determinant
            return d
      end
Step 2: Calculate Pivot Block
      Pivot = det(A(1:4, 1:4)) // 4×4
determinant
      if Pivot == 0
            Interchange rows to make
Pivot ≠ 0
            Pivot = det(A(1:4, 1:4)) //
recompute
      end
Step 3: Build Condensed Matrix B
      Initialize B = zeros(m-4, n-4)
        for i = 5 to m
          for j = 5 to n
            B(i-4, j-4) =
det(A([1,2,3,4,i], [1,2,3,4,j])) // 5×5
determinant
          end
        end
Step 4: Compute Subtracted Columns
      Sub1 = A(:,2) - A(:,1)    // m×1
      Sub2 = A(:,4) - A(:,3)    // m×1
Step 5: Recursive Chio4 Calls
      T1 = det_Chio4(B)
      T2 = det_Chio4([A(:,1:2) Sub2
A(:,5:n)]) // m×(n-1)
      T3 = det_Chio4([Sub1 Sub2
A(:,5:n)]) // m×(n-1)
      T4 = det_Chio4([Sub1 A(:,3:n)]) //
m×(n-2)
Step 6: Combine Determinants
      d = T1 / Pivot^(m-5) + T2 + T3 +
(-1)^m * T4
return d
```

In the following, the time complexity of the above pseudocode is calculated.

```
Step 1: Edge Case Checks
if m > n or m == n or m < 5:
```
\\Radic_Definition(A)

- $Radic\_Definition(A)$ for an m×n matrix has exponential cost based on combinations of n choose m, leading to approximately $O(n^m)$ in the worst case.
- However, this step is only executed once, so it's treated as $O(1)$ or constant for large inputs (early exit).

```
Step 2: Compute Pivot Block (4×4
determinant)
Pivot = det(A(1:4, 1:4))
```

- Time Complexity: Computing a 4×4 determinant has constant cost: $O(1)$.
- If the pivot is zero, rows are interchanged and determinant is recomputed: still $O(1)$.

```
Step 3: Build Condensed Matrix B
for i = 5 to m:
    for j = 5 to n:
      B(i-4, j-4) = det(A([1,2,3,4,i],
[1,2,3,4,j])) // 5×5 determinant
```

- Number of iterations: $(m-4) \times (n-4)$
- Each iteration involves a 5×5 determinant $O(1)$
- Total Cost: $O((m-4)(n-4)) \approx O(mn)$ (linear growth based on input size, ignoring constants)

```
Step 4: Subtracted Columns
Sub1 = A(:,2) - A(:,1)
Sub2 = A(:,4) - A(:,3)
```

- Subtractions on column vectors of size m: each is $O(m)$
- Total Cost: $O(m)$

```
Step 5: Recursive Calls
T1 = det_Chio4(B)                    //
size: (m-4) × (n-4)
T2 = det_Chio4([A(:,1:2) Sub2 A(:,5:n)])
// size: m × (n-1)
T3 = det_Chio4([Sub1 Sub2 A(:,5:n)])
// size: m × (n-1)
T4 = det_Chio4([Sub1 A(:,3:n)])
// size: m × (n-2)
```

- Each det_Chio4 call costs approximately $O(m^2 \cdot n \cdot (n-m))$, assuming worst-case recursion and submatrix evaluation.

```
Step 6: Combine Determinants
d = T1 / Pivot^(m-5) + T2 + T3 + (-1)^m * T4
```

- Simple arithmetic operations on scalar values: $O(1)$
- The dominant cost comes from the recursive det_Chio4 calls on matrices of size near $m \times n$. Each has complexity:
- $O(m^2 \cdot n \cdot (n-m))$.

Even though the time complexity resulted the same as previous papers, but T1's size is much smaller: $(m-4) \times (n-4)$. So:

$$T1: O((m-4)^2 \cdot (n-4) \cdot (n-m)) \approx O(m^2 \cdot n \cdot (n-m))$$

- $T2, T3, T4$: *each also* $\approx O(m^2 \cdot n \cdot (n-m))$
- Since there are 4 recursive calls:
- Total Time Complexity: $O(m^2 \cdot n \cdot (n-m))$

In the following we have tested the execution time of the algorithm presented above and compared with the execution time of algorithms based on Theorem 1 (Algorithm 2), and Theorem 2 Algorithm 3). We simulated the execution time for different order of non-square determinants at the same time with three different algorithms, this simulation is performed in MATLAB 2021b version environment, in Asus ROG Flow Z13, with 12th Gen Intel(R) Core(TM) i7-12700H, 16.0 GB of RAM, NVIDIA GeForce RTX 3050 GPU.

TABLE I. NON-SQUARE DETERMINANT CALCULATION EXECUTION TIME SIMULATION

Order	Algorithm 2	Algorithm 3	Algorithm4	1-2	1-3	2-3	1/2-1	1/3-1	2/3-1
	1	2	3						
6x10	0.00174	0.00147	0.00142	0.00027	0.00032	0.00005	18.22%	22.29%	3.45%
6x15	0.02364	0.01947	0.01890	0.00417	0.00474	0.00057	21.43%	25.06%	2.99%
6x20	0.11774	0.09827	0.09539	0.01948	0.02236	0.00288	19.82%	23.44%	3.02%
6x25	0.60247	0.50821	0.49373	0.09426	0.10875	0.01449	18.55%	22.03%	2.93%
6x30	2.07658	1.74937	1.69666	0.32721	0.37992	0.05271	18.70%	22.39%	3.11%
6x35	5.94749	4.94864	4.76868	0.99885	1.17881	0.17996	20.18%	24.72%	3.77%
6x40	13.60411	11.13902	10.82880	2.46509	2.77531	0.31021	22.13%	25.63%	2.86%
10x15	0.02089	0.01748	0.01690	0.00342	0.00400	0.00058	19.56%	23.64%	3.41%
10x20	0.66432	0.54791	0.52904	0.11641	0.13528	0.01887	21.25%	25.57%	3.57%
10x25	11.82490	9.71338	9.40368	2.11152	2.42122	0.30970	21.74%	25.75%	3.29%
10x30	109.36627	90.52638	87.65362	18.83989	21.71265	2.87276	20.81%	24.77%	3.28%
15x20	0.07646	0.06421	0.06220	0.01225	0.01425	0.00200	19.08%	22.91%	3.22%
15x25	16.16055	13.34079	12.92512	2.81977	3.23544	0.41567	21.14%	25.03%	3.22%
20x25	0.53221	0.43348	0.41769	0.09873	0.11452	0.01579	22.78%	27.42%	3.78%
25x30	1.84497	1.53064	1.48256	0.31433	0.36241	0.04808	20.54%	24.44%	3.24%
30x35	5.07792	4.12053	4.00130	0.95739	1.07663	0.11924	23.23%	26.91%	2.98%
35x40	11.60587	9.65014	9.38158	1.95573	2.22429	0.26856	20.27%	23.71%	2.86%
40x45	24.90592	20.41070	19.80006	4.49522	5.10586	0.61064	22.02%	25.79%	3.08%
45x50	47.29763	39.12098	37.99821	8.17664	9.29942	1.12277	20.90%	24.47%	2.95%
50x55	70.00049	57.50784	55.47739	12.49264	14.52310	2.03046	21.72%	26.18%	3.66%
55x60	105.70073	87.41192	84.88040	18.28881	20.82033	2.53152	20.92%	24.53%	2.98%
60x65	159.60811	132.86612	127.32060	26.74199	32.28751	5.54552	20.13%	25.36%	4.36%
65x70	241.00824	201.95651	194.80052	39.05174	46.20772	7.15598	19.34%	23.72%	3.67%
70x75	363.92245	305.97389	297.04480	57.94856	66.87765	8.92909	18.94%	22.51%	3.01%
75x80	549.52290	466.60031	453.04764	82.92259	96.47525	13.55266	17.77%	21.29%	2.99%
80x85	845.27481	706.56647	684.07242	138.70834	161.20239	22.49405	19.63%	23.57%	3.29%
85x90	1287.91221	1089.84970	1056.10863	198.06251	231.80358	33.74107	18.17%	21.95%	3.19%
90x95	1901.86832	1589.77455	1539.16294	312.09377	362.70537	50.61161	19.63%	23.57%	3.29%
95x100	2882.80247	2384.66183	2318.74442	498.14065	564.05806	65.91741	20.89%	24.33%	2.84%
100x105	4379.20371	3576.99274	3483.11663	802.21097	896.08708	93.87611	22.43%	25.73%	2.70%
105x110	6428.80556	5375.48911	5204.67494	1053.31646	1224.13063	170.81417	19.59%	23.52%	3.28%
110x115	9678.20835	8068.23366	7822.01241	1609.97469	1856.19594	246.22125	19.95%	23.73%	3.15%
Average improvement							20.36%	24.25%	3.23%

The following is presented the graphical view of the comparison presented on table 1:

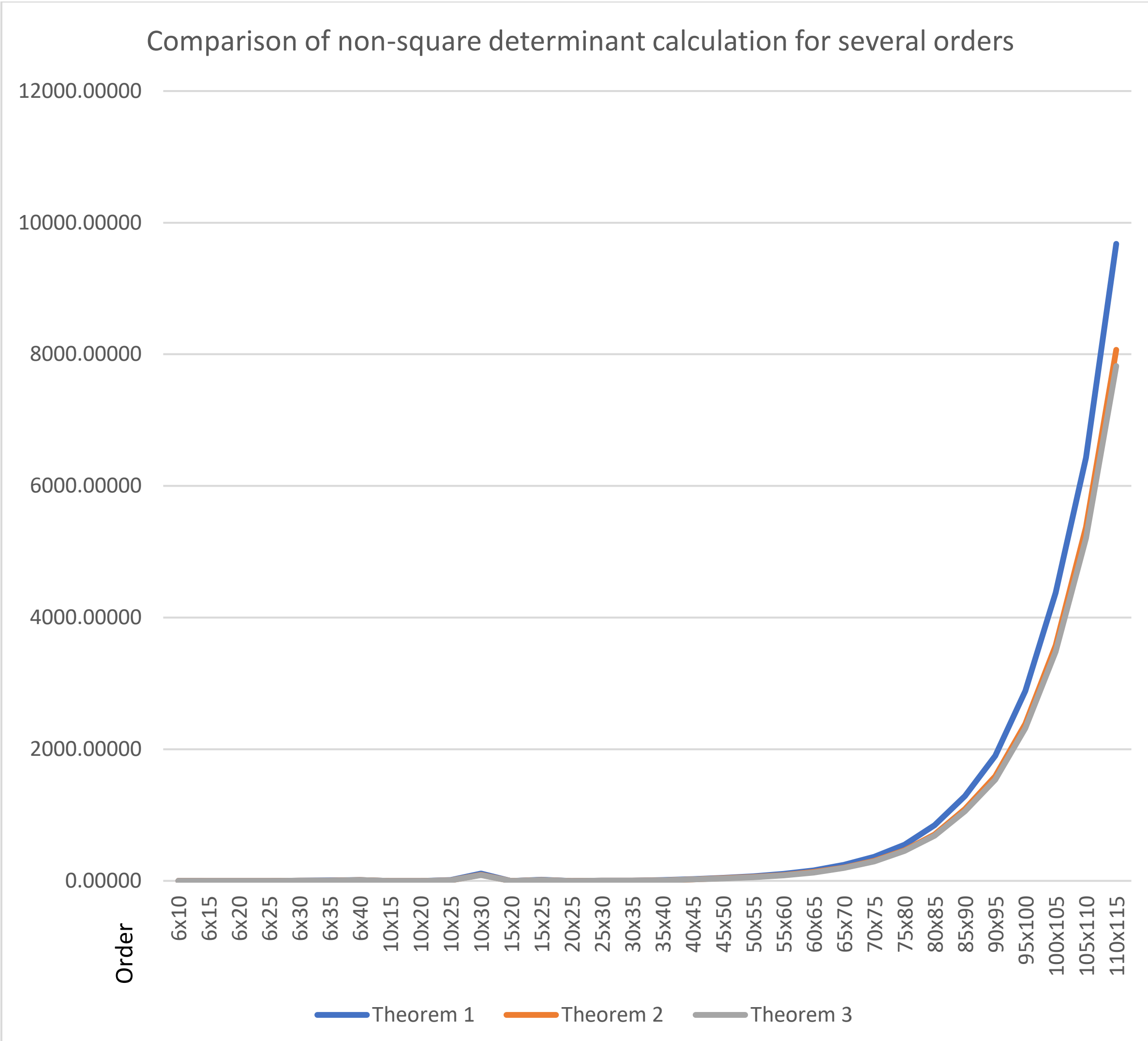

Fig 1. Graphical view of the comparison presented on Table I.

Results of the simulation are presented in seconds the following Table I. The simulations show that the enhanced Chio-Like method in Theorem 3 (Algorithm 4) has improved the processing speed for about 3.2% compared to the Chio-Like method in Theorem 2 (Algorithm 3) and about 24.3% compared to Chio-Like method in Theorem 1 (Algorithm 2).

IV. Conclusion

This paper modifies the Chio-like method for computing non-square determinants. We reduce four rows and columns to calculate third-order blocks, using a fourth-order determinant as the pivot. A computer algorithm implementing Theorem 3 is presented, and we compare the execution times of Algorithms 2, 3, with the new approach (Theorem 3 and Algorithm 4) by analyzing the asymptotic time complexity of the new method.

MATLAB was used to generate random determinants and compare the three algorithms; results are shown in Table 1 and Figure 1. From the simulation it is noted an improvement of about 24.5% compared to the Chio's-like methodology definition according Algorithm 1, and an improvement of about 3.2% compared to the Chio's-like methodology definition according Algorithm 2. This advancement can further find application in multimedia retrieval for more efficient calculation of similarity scores that are essential in multimedia comparison in the retrieval process.

References

[1] A., Kaur, T., & Kaur, S. (2024). Content-Based Video and Image Retrieval in the Modern Era: Apprehensions and Scope. In Latest Trends in Engineering and Technology (pp. 141-145). CRC Press. URL: https://www.taylorfrancis.com/chapters/edit/10.1201/9781032665443-

20/content-based-video-image-retrieval-modern-era-apprehensions-scope-amarjeet-kaur-taranjeet-kaur-sarabpreet-kaur

[2] Zhang, J., Wu, Y., Hao, F., Liu, X., Li, M., Zhou, D., & Zheng, W. (2024). Double similarities weighted multi-instance learning kernel and its application. Expert Systems with Applications, 238, 121900. URL: https://doi.org/10.1016/j.eswa.2023.121900

[3] Kang, Z., Peng, C., & Cheng, Q. (2017). Kernel-driven similarity Learning. Neurocomputing, 267, 210-219. URL: https://doi.org/10.1016/j.neucom.2017.06.005

[4] Gwashavanhu, B. K., Oberholster, A. J., & Heyns, S. P. (2024). A comparative study of principal component analysis and kernel principal component analysis for photogrammetric shape-based turbine blade damage analysis. Engineering Structures, 318, 118712. [4] P.H. Gosselin, M. Cord, S. URL: https://doi.org/10.1016/j.engstruct.2024.118712

[5] Bach, F. (2022). Information theory with kernel methods. IEEE Transactions on Information Theory, 69(2), 752-775. URL: 10.1109/TIT.2022.3211077

[6] Batlle, P., Darcy, M., Hosseini, B., & Owhadi, H. (2024). Kernel methods are competitive for operator learning. Journal of Computational Physics, 496, 112549. URL: https://doi.org/10.1016/j.jcp.2023.112549

[7] Rasekhinezhad, H., Abbasbandy, S., Allahviranloo, T., & Baboliand, E. (2024). Applications of new smart algorithm based on kernel method for variable fractional functional boundary value problems. International Journal of Dynamics and Control, 12(8), 2795-2802. URL: https://doi.org/10.1007/s40435-024-01397-5

[8] Ghojogh, B., Ghodsi, A., Karray, F., & Crowley, M. (2021). Reproducing Kernel Hilbert Space, Mercer's Theorem, Eigenfunctions, Nystr\" om Method, and Use of Kernels in Machine Learning: Tutorial and Survey. arXiv preprint arXiv:2106.08443. URL: https://doi.org/10.48550/arXiv.2106.08443

[9] Liu, X., Zhang, Z., Igathinathane, C., Flores, P., Zhang, M., Li, H., ... & Kim, H. J. (2024). Infield corn kernel detection using image processing, machine learning, and deep learning methodologies under natural lighting. Expert Systems with Applications, 238, 122278. URL: https://doi.org/10.1016/j.eswa.2023.122278

[10] Croughan, M. K., Paganin, D. M., Alloo, S. J., Ahlers, J. N., How, Y. Y., Harker, S. A., & Morgan, K. S. (2024). Correcting directional dark field x-ray imaging artefacts using position dependent image deblurring and attenuation removal. Scientific Reports, 14(1), 17807. URL: https://doi.org/10.1038/s41598-024-68659-2

[11] Arora, P., Mehta, R., & Ahuja, R. (2024). An integration of meta-heuristic approach utilizing kernel principal component analysis for multimodal medical image registration. Cluster Computing, 27(5), 6223-6246. URL: https://doi.org/10.1007/s10586-024-04281-1

[12] Dai, M., Raffiee, A. H., Jain, A., & Correa, J. (2024). Evaluating Transferability in Retrieval Tasks: An Approach Using MMD and Kernel Methods. In Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition (pp. 22390-22400). URL: 10.1109/CVPR52733.2024.02113

[13] Yi, Q., He, Y., Wang, J., Song, X., Qian, S., Yuan, X., ... & Shi, T. (2025). Score: Story coherence and retrieval enhancement for ai narratives. arXiv preprint arXiv:2503.23512. URL: https://doi.org/10.48550/arXiv.2503.23512

[14] Cullis, E.C. (1913). Matrices and determinoids. Cambridge: Cambridge: University Press. URL: https://doi.org/10.2307/3603184

[15] Radic, M. (1966). Definition of Determinant of Rectangular Matrix, Glasnik Matematicki, 17-22. URL: https://web.math.pmf.unizg.hr/glasnik/Vol/vol01no1.html

[16] Salihu, A., & Marevci, F. (2021). Chio's-like method for calculating the rectangular (non-square) determinants: Computer algorithm interpretation and comparison. European Journal of Pure and Applied Mathematics, 14(2), 431-450. URL: https://doi.org/10.29020/nybg.ejpam.v14i2.3920

[17] Salihu, A., Snopce, H., Luma, A., & Ajdari, J. (2023). MODIFIED CHIOS-LIKE METHOD FOR RECTANGULAR DETERMINANT CALCULATIONS. Advanced Mathematical Models & Applications, 8(3). URL: https://jomardpublishing.com/UploadFiles/Files/journals/AMMAV1N1/V8N3/Salihu_et_al.pdf

[18] Salihu, A., & Marevci, F. (2019). Determinants order decrease/increase for k orders, interpretation with computer algorithms and comparison. Computer Science, 9(2), 501-518. URL: https://future-in-tech.net/14.2/R-Marecvi-Salihu.pdf

[19] Makarewicz, A., & Pikuta, P. (2020). Cullis-Radić determinant of a rectangular matrix which has a number of identical columns. Annales Universitatis Mariae Curie-Sklodowska, sectio A–Mathematica, 74(2), 41-60. URL: https://bibliotekanauki.pl/articles/1395917.pdf

[20] Makarewicz, A., Pikuta, P., & Szałkowski, D. (2014). Properties of the determinant of a rectangular matrix. Annales Universitatis Mariae Curie-Skłodowska, sectio A–Mathematica, 68(1). URL: https://bibliotekanauki.pl/articles/747308.pdf